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Nonlinear Classes of Splines and Variational Problems

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1. INTRODUCTION

The purpose of this paper is to examine certain constrained minimization problems related to best interpolation in the space $L_\infty{}^m[a, b] = \{f \in AC^{m-1}[a, b] : f^{(m)} \in L_\infty[a, b]\}$ whose solutions in various special cases are classes of (nonlinear) splines. To be more specific, suppose $\{\lambda_i\}_1^n$ is a set of n linearly independent bounded linear functionals on $L_\infty{}^m[a, b]$, and that f_0 is a prescribed element of $L_\infty{}^m[a, b]$. We define

$$U = \{f \in L_\infty{}^m[a, b]: \lambda_i f = \lambda_i f_0\,,\ i = 1, 2,..., n\}. \tag{1.1}$$

This is the set of all functions in $L_\infty{}^m[a, b]$ interpolating f_0 with respect to $\{\lambda_i\}_1^n$. To define a smoothest such interpolate, suppose L is a mapping of $L_\infty{}^m[a, b]$ into $L_\infty[a, b]$, and that ρ is a (possibly nonlinear) functional on $L_\infty[a, b]$. We seek $s \in U$ such that

$$\alpha = \rho(Ls) = \inf_{u \in U} \rho(Lu). \tag{1.2}$$

A solution of (1.2) will be called a *spline function* interpolating f_0 with respect to $\{\lambda_i\}_1^n$.

Problems of the form (1.2) have been intensively studied in the case where L is a linear differential operator of order m and ρ is the essential supremum

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norm on $L_\infty[a, b]$ (e.g., see [3–7, 9, 10, 13, 14, 16, 21]). There are also results for the case where L is allowed to be nonlinear; see [8, 9, 13, 14].

Recently, the first-named author (see [1, 2]) discovered that several classes of nonlinear splines (including, for example, rational, exponential, and logarithmic splines) which previously were defined only constructively (see [19, 20] for the rational case) also satisfy a best interpolation property of the form (1.2) with $m = 2$, $L = D^2$, and ρ defined by an appropriate convex integral. The methods and results of convex analysis (such as in [12, 17, 18]) were the basic tools. Our aim here is to use the same tools to carry out the analysis of (1.2) for a wide class of m, L, U, and ρ.

2. Assumptions

In order to apply the methods of convex programming to (1.2), we have to make some assumptions on L, $\{\lambda_i\}_1^n$, and ρ. First, we suppose L is an mth order nonsingular linear differential operator of the form

$$L = \sum_{i=0}^{m} a_i D^i, \qquad a_i \in L_1{}^i[a, b],\ i = 0, \ldots, m,\ a_m(x) > 0 \text{ for } x \in [a, b]. \tag{2.1}$$

It is well known (cf. [13, 15]) that L maps $L_\infty{}^m[a, b]$ onto $L_\infty[a, b]$ and that corresponding to L there is a Green's function $g(x, y)$ such that for every $f \in L_\infty{}^m[a, b]$

$$f(x) = p_f(x) + \int_a^b g(x, y)\, Lf(y)\, dy, \tag{2.2}$$

where p_f is the element in $N_L = \{f \in L_\infty{}^m[a, b] : Lf = 0\}$ with $p_f^{(j)}(a) = f^{(j)}(a)$, $j = 0, 1, \ldots, m - 1$. Specifically, $g(x, y)$ can be constructed in the form

$$\begin{aligned} g(x, y) &= \sum_{i=1}^{m} u_i(x)\, u_i{}^*(y), \qquad && x \geqslant y, \\ &= 0, && x < y, \end{aligned} \tag{2.3}$$

where $\{u_i\}_1^m$ span N_L and $\{u_i{}^*\}_1^m$ span N_{L^*} with L^* the formal adjoint of L.

Concerning $\Lambda = \operatorname{span}\{\lambda\}_1^n$, we suppose that it is total over N_L (i.e., $\lambda_i p = 0$, $i = 1, 2, \ldots, n$, and $p \in N_L$ implies $p = 0$), and that $\lambda_i g(\cdot,\ \ ) \in L_1[a, b]$, $i = 1, 2, \ldots, n$. The totality assumption is satisfied whenever Λ contains enough point evaluation functionals, for example, and the second assumption is satisfied for broad classes of linear functionals, including the extended Hermite–Birkhoff linear functionals which are defined as linear combinations of point evaluators of derivatives up to order $m - 1$ (see [13, 15]).

We shall consider ρ defined by certain convex integrals. Let

$$\mathscr{F} = \{F : \mathbb{R} \to \mathbb{R} \cup \{\infty\} : F \text{ is convex, lower semicontinuous, } D_F \neq \varnothing, F \in C^2(D_F), \text{ and } F' \text{ is strictly monotone increasing on } D_F\}, \tag{2.4}$$

where

$$D_F = \text{interior } \{x \in \mathbb{R} : F(x) < \infty\}. \tag{2.5}$$

For some properties and examples of functions in $\mathscr{F}$, see [1, 2]. Given $F \in \mathscr{F}$, we define a corresponding functional on $L_\infty[a, b]$ by the convex integral

$$\rho_F(g) = \int_a^b F(g(t))\, dt. \tag{2.6}$$

3. Equivalent Problems

It will be convenient to reformulate the minimization problem (1.2) in the space $L_\infty[a, b]$, and then to convert it to a dual problem. First, we have

Lemma 3.1. *Let L, Λ, and ρ be as in Section 2. Suppose U is defined as in* (1.1). *Then there exist $\{\tilde{\lambda}_i\}_1^n \subset \Lambda$ such that $LU = V$, where*

$$V = \left\{v \in L_\infty[a, b] : \int_a^b v(t)\, h_i(t)\, dt = \int_a^b Lf_0(t)\, h_i(t)\, dt,\ i = m + 1, \ldots, n\right\}, \tag{3.1}$$

and $h_i(t) = \tilde{\lambda}_i\, g(\cdot, t)$, $i = m + 1, \ldots, n$. Moreover, s will be a solution of (1.2) *if and only if $\sigma = Ls$ is a solution of*

$$\alpha = \rho_F(\sigma) = \inf_{v \in V} \rho_F(v). \tag{3.2}$$

Proof. By the assumption that Λ is total over N_L and the fact that N_L is m-dimensional, there exist $\{\tilde{\lambda}_i\}_1^m$ which are linearly independent over N_L. Now let $\{\tilde{\lambda}_i\}_{m+1}^n \subset \Lambda$ be chosen so that $\{\tilde{\lambda}_i\}_1^n$ span Λ and $\tilde{\lambda}_i\, p = 0$, all $p \in N_L$ and all $i = m + 1, \ldots, n$. (There are several ways to construct $\{\tilde{\lambda}_i\}_{m+1}^n$, although their span is uniquely determined. For example, we may take $\tilde{\lambda}_i = \lambda_i - \sum_{j=1}^m c_{ij} \tilde{\lambda}_j$, with coefficients chosen so that $\tilde{\lambda}_i u_j = 0, j = 1, 2, \ldots, m$.) Now applying $\tilde{\lambda}_i$ to the generalized Taylor formula (2.2), we have

$$\tilde{\lambda}_i f = \tilde{\lambda}_i \int_a^b g(\cdot, y)\, Lf(y)\, dy = \int_a^b Lf(y)\, h_i(y)\, dy, \qquad i = m + 1, \ldots, n.$$

(Note: $\tilde{\lambda}_i\, g$ is integrable by the assumptions on Λ and the definition of the $\tilde{\lambda}$'s.)

Now, if $u \in U$, than $\tilde{\lambda}_i u = \int_a^b Lu(y)\, h_i(y)\, dy = \tilde{\lambda}_i f_0 = \int_a^b Lf_0(y)\, h_i(y)\, dy$, $i = m+1,..., n$; i.e., $Lu \in V$, and so $LU \subset V$.

Conversely, if $v \in V$, let $p \in N_L$ be chosen such that

$$\tilde{\lambda}_i p = \tilde{\lambda}_i f_0 - \int_a^b v(y)\, h_i(y)\, dy, \qquad i = 1, 2,..., m.$$

Then

$$u(x) = p(x) + \int_a^b v(y)\, g(x, y)\, dy \in U,$$

and $Lu = v$. This proves $V \subset LU$ and the lemma is established. ∎

Our next task is to dualize problem (3.2). Let

$$V_0 = \left\{ v \in L_\infty[a, b]: \int_a^b v(y)\, h_i(y)\, dy = 0, i = m+1,..., n \right\}. \tag{3.3}$$

Then $V_0^\perp = \{h \in L_1[a, b] : \int_a^b v(y)\, h(y)\, dy = 0\} = \text{span}\, \{h_i\}_{m+1}^n$. We also need some notation from the theory of convex analysis. Let X be a locally convex space. For each $y \in X^*$ we shall write $\langle x, y \rangle$ for the value of the linear functional y operating on x. Now if φ is an extended real-valued function defined on X, we define its convex conjugate φ^* by

$$\varphi^*(y) = \sup_{x \in X} (\langle x, y \rangle - \varphi(x)), \qquad y \in X^* \tag{3.4}$$

and its concave conjugate by

$$\varphi^+(y) = \inf_{x \in X} (\langle x, y \rangle - \varphi(x)), \qquad y \in X^*. \tag{3.5}$$

Finally, we need to introduce the set

$$C(I; D_F) = \{f \in C[I]: f(t) \in D_F \text{ for all } t \in I\}. \tag{3.6}$$

LEMMA 3.2. *Suppose $V \cap C(I; D_F) \neq \varnothing$, where we write $I = [a, b]$. Then the value α of the infimum in* (1.2) *is also given by*

$$\alpha = \max_{z \in R^{n-m}} \Psi(z), \tag{3.7}$$

where

$$\Psi(z) = \sum_{i=1}^{n-m} z_i \tilde{\lambda}_{i+m} f_0 - \rho_{F^*}\left(\sum_{i=1}^{n-m} z_i h_{i+m} \right). \tag{3.8}$$

Proof. If $v \in V \cap C(I; D_F)$, then by [18, Theorem 2], ρ_F is continuous at v. Hence, by the result of [12, p. 68],

$$\inf_{v \in V} \rho_F(v) = \max_{g \in L_\infty(I)^*} ((-\delta_V)^+(g) - \rho_F^*(g)),$$

where ρ_F^* is the convex conjugate of ρ_F, $(-\delta_V)^+$ is the concave conjugate of $-\delta_V$, and

$$\begin{aligned} \delta_V(f) &= 0, & f &\in V, \\ &= \infty, & f &\notin V. \end{aligned}$$

But

$$\begin{aligned} (-\delta_V)^+(g) &= \int_a^b g(t)\, Lf_0(t)\, dt, & g &\in V_0^\perp, \\ &= -\infty, & g &\notin V_0^\perp, \end{aligned}$$

and

$$\rho_F^* \,|_{L_1(I)} = \rho_{F^*}$$

(See [18, Theorem 1]). Since $V_0^\perp$ is spanned by $\{h_i\}_{m+1}^n$,

$$\int_a^b g(t)\, Lf_0(t)\, dt = \sum_{i=1}^{n-m} z_i \tilde{\lambda}_{i+m} f_0, \qquad \text{if} \quad g = \sum_{i=1}^{n-m} z_i h_{i+m}.$$

Substituting in the above yields (3.7). ▮

4. Existence and Abstract Characterization

Before giving sufficient conditions for the existence of solutions of (1.2) (or, equivalently, of (3.2)), we first give an abstract characterization of solutions of (3.2) which are sufficiently smooth.

THEOREM 4.1. *A function $\sigma \in V \cap C(I; D_F)$ is a solution of* (3.2) *if and only if*

$$F' \circ \sigma \in V_0^\perp. \tag{4.1}$$

Proof. We need the concept of a subdifferential (see, e.g., [12]). If φ maps a locally convex space X into $\mathbb{R} \cup \{\infty\}$ and is convex then $\partial\varphi(x_0) = \{\lambda \in X^* : \lambda(x - x_0) \leqslant \lambda(x) - \lambda(x_0), \text{ all } x \in X\}$ is called the ***subdifferential*** of φ at $x_0 \in X$. For any $\sigma \in C(I; D_F)$, $\partial\rho_F(\sigma) = \{F' \circ \sigma\}$ by [18, Corollary 2C]. Moreover, since $\lambda \in L_\infty^*(I)$ can satisfy $\lambda(g - v) \leqslant \delta_V(g) - \delta_V(v)$ for fixed $v \in V$ and all $g \in L_\infty(I)$ if and only if $\lambda \in V_0^\perp$, we conclude that $\partial\delta_V(v) = V_0^\perp$.

If (4.1) holds then $\partial\rho_F(\sigma) \cap \partial(-\delta_V(\sigma)) \neq \varnothing$ as it contains $F' \circ \sigma$. By the result of [12, pp. 68–69] (recall that ρ_F is continuous at σ under the hypotheses here as in Lemma 3.2), it follows that σ is a solution of (3.2).

Conversely, if σ is a solution of (3.2), then again by [12, pp. 68–69], $\partial\rho_F(\sigma) \cap \partial(-\delta_V(\sigma)) \neq \varnothing$. But the first set here is $F' \circ \sigma$ as noted above, while the second is $V_0^{\perp}$. We conclude that (4.1) must hold. ∎

In the next theorem we establish existence of solutions of (2.1) under certain additional conditions on F and Λ.

THEOREM 4.2. *Suppose*

$$V \cap C(I; D_F) \neq \varnothing, \tag{4.2}$$

$$V_0^{\perp} \cap \operatorname{dom}(\rho_{F^*}) \subset V_0^{\perp} \cap C(I; F'D_F), \tag{4.3}$$

where $F'D_F = \{w : w = F'x, x \in D_F\}$ *is the image of* D_F *under* F' *and* dom $(\rho_{F*}) = \{ g \in L_1(I) \colon \rho_{F*}(g) < \infty\}$. *Then there exists at least one solution* σ *of* (3.2), *and hence at least one solution of* (1.2).

Proof. Let $z^* \in \mathbb{R}^{n-m}$ be a vector which attains the maximum in (3.7). Then by elementary calculus, z^* must satisfy the system of equations

$$(\partial\Psi/\partial z_j)(z) = (\partial/\partial z_j)\left[\sum_{i=1}^{n-m} z_i\tilde{\lambda}_{i+m}f_0 - \rho_{F^*}\left(\sum_{i=1}^{n-m} z_i h_{i+m}\right)\right] = 0,$$

$j = 1, 2,\ldots, n - m$. This is simply the system

$$\int_a^b F^{*\prime}\left(\sum_{i=1}^{n-m} z_i^* h_{i+m}(t)\right) h_{j+m}(t)\,dt = \tilde{\lambda}_{j+m}f_0, \qquad j = 1, 2,\ldots, n - m.$$

Moreover, since for $F \in \mathscr{F}$ we have $F^{*\prime} = (F')^{-1}$ (see [2, p. 1; 1, p. 24]), this is also equivalent to the system

$$\int_a^b F'^{-1}\left(\sum_{i=1}^{n-m} z_i^* h_{i+m}(t)\right) h_{j+m}(t)\,dt = \tilde{\lambda}_{j+m}f_0, \tag{4.4}$$

$j = 1, 2,\ldots, n - m$. Since $g_{z^*} = \sum_{i=1}^{n-m} z_i^* h_{i+m}$ certainly belongs to dom (ρ_{F^*}) (as otherwise $-\rho_{F^*}(g_{z^*})$ would be $-\infty$ and z^* would not maximize (3.7)), hypothesis (4.3) implies $g_{z^*} \in V_0^{\perp} \cap C(I; F'D_F)$. But then $\sigma = F'^{-1}(g_{z^*}) \in C(I; D_F)$, and $F' \circ \sigma = g_{z^*} \in V_0^{\perp}$. Moreover, in view of (4.4), it is clear that $\sigma \in V$. Now Theorem 4.1 asserts that σ is a solution of (3.2). ∎

System (4.4) is a nonlinear system of $n - m$ equations which can be used numerically for the determination of the vector z^*. This vector can also be computed numerically by attacking the dual problem in Lemma 3.2 directly. Indeed, it suffices to seek a maximum of $\psi(z)$ as defined in (3.8) over the set

$K = \{z \in \mathbb{R}^{n-m}: g_z = \sum_{i=1}^{n-m} z_i h_{i+m} \in C(I: F'D_F)\}$. Since this set is convex while ψ is concave and twice differentiable, standard gradient methods are applicable (see [1, 2]). When Ls is sufficiently smooth, the following result shows that system (4.4) is also a necessary condition for s to be a solution of (2.1).

THEOREM 4.3. *If s is a solution of* (1.2) *and $Ls \in C(I; D_F)$, then $g = F' \circ Ls$ satisfies* (4.4).

Proof. Since $s \in U$, we have

$$\int_a^b F^{-1}(g(t))\, h_{i+m}(t)\, dt = \int_a^b Ls(t)\, h_{i+m}(t)\, dt$$
$$= \tilde{\lambda}_{i+m} s = \tilde{\lambda}_{i+m} f_0, \qquad i = 1, 2,..., n-m. \quad \blacksquare$$

Generally, hypothesis (4.3) is easily verified while hypothesis (4.2) is not (see [1, 2]). We also note that the hypothesis that $Ls \in C(I; D_F)$ in Theorem 4.3 cannot be removed, as there are examples of problem (1.2) which possess a unique solution s with $Ls \notin C(I; D_F)$; see [1, p. 38]. On the other hand, if (4.2) and (4.3) hold, then Theorem 4.2 establishes the existence of at least one solution with $Ls \in C(I; D_F)$.

5. STRUCTURAL CHARACTERIZATION

For most constrained variational problems with spline function solutions, it is possible to give detailed structural characterizations of the splines (cf., e.g., [11, 14, 15]), at least for nice classes of Λ. Here we give just one such characterization result for the case where Λ consists of Hermite–Birkhoff linear functionals.

THEOREM 5.1. *Let $a \leqslant x_1 < \cdots < x_k \leqslant b$, and suppose $\Lambda = \{\lambda_{ij} = e_{x_i}^{\nu_{ij}}\}_{j=1,i=1}^{\ell_i,\ k}$, where $1 \leqslant \ell_i \leqslant m$ and $0 \leqslant \nu_{i1} < \cdots < \nu_{i\ell_i} \leqslant m-1$ are given integers, $i = 1, 2,..., k$ (and where e_x^ν denotes the point evaluator of the νth derivative at x;* i.e., *$e_x^\nu f = f^{(\nu)}(x)$). Λ is called a Hermite–Birkhoff set of linear functionals. Suppose Λ is total over N_L. Given $f_0 \in L_\infty^m[a, b]$, let U be defined by* (1.1). *Let $s \in U$ be a spline interpolating U with respect to Λ,* i.e., *a solution of* (1.2). *Then, if $Ls \in C(I; D_F)$, there exist functions $\alpha_i \in N_{L^*}$, $i = 1, 2,..., k-1$, such that*

$$F'(Ls(t)) = \alpha_i(t) \qquad \text{a.e. on } (x_i, x_{i+1}),\ i = 1, 2,..., k-1; \tag{5.1}$$

$$F'(Ls(t)) = 0 \qquad \text{a.e. on } (a, x_1) \text{ and } (x_k, b); \tag{5.2}$$

$$\text{jump}[D^\nu F'(Ls)]_{x_i} = 0, \qquad \text{all } \nu \in \{0, 1,..., m-1\} \backslash \{\nu_{i1},..., \nu_{i\ell_i}\},\ i = 1, 2,..., k. \tag{5.3}$$

(*Here* $\operatorname{jump}[\varphi]_t = \varphi(t+) - \varphi(t-)$ *if* $a < t < b$, *and* $\operatorname{jump}[\varphi]_a = \varphi(a+)$ *while* $\operatorname{jump}[\varphi]_b = -\varphi(b-)$.)

Proof. To prove (5.1), let $J = (x_i, x_{i+1})$ with $1 \leqslant i \leqslant k-1$, and let $\varphi \in C_c^\infty(J)$. Then

$$\begin{aligned} v(x) &= L\varphi(x), && x \in J, \\ &= 0, && \text{otherwise,} \end{aligned}$$

clearly belongs to V_0. Hence, by Theorem 4.1, we must have

$$\int_J F'(Ls(t))\, L\varphi(t)\, dt = 0.$$

Since φ was an arbitrary $C_c^\infty(J)$ function, familiar arguments (cf., e.g., [15]) imply (5.1) on J. If $J = (a, x_1)$, say, then we can take $\varphi \in \{C^\infty(J) : \varphi^{(j)}(x_1) = 0, j = 0, 1,..., m-1\}$. Then v as defined above again belongs to V_0, and this time (5.2) follows.

The proof of (5.3) is a repeat of the proofs used for Lg-splines (as in [11, 15]). In particular, given ϵ sufficiently small that $(x_i - \epsilon, x_i + \epsilon)$ contains no other knots, let $\varphi \in C_c^\infty(x_i - \epsilon, x_i + \epsilon)$ with $\varphi^{(j)}(x_i) = \delta_{\nu j}$, where ν is fixed in $\{0, 1,..., m-1\} \backslash \{\nu_{i1},..., \nu_{il_i}\}$. Then v as defined above again belongs to V_0, and we obtain

$$\int_{x_i-\epsilon}^{x_i} F'(Ls(t))\, L\varphi(t)\, dt + \int_{x_i}^{x_i+\epsilon} F'(Ls(t))\, L\varphi(t)\, dt = 0.$$

Integrating by parts and using (5.1) and (5.2) (cf. [15]), we obtain

$$\sum_{j=0}^{m-1} \varphi^{(j)}(x_i) \operatorname{jump}[D^j F'(Ls)]_{x_i} = 0,$$

and (5.3) follows. ∎

A similar characterization theorem holds for extended Hermite–Birkhoff linear functionals (cf. [15]).

6. Uniqueness

In this section we discuss the uniqueness of solutions of problem (1.2). In view of the assumption that Λ is total over N_L, the uniqueness of solutions of (1.2) is equivalent to uniqueness in problem (3.2). As with our discussion of existence in Section 4, it will be convenient to examine the dual problem (3.7).

LEMMA 6.1. *Let $F \in \mathscr{F}$, and suppose that* (4.2) *and* (4.3) *hold. Then* (1.2) *has a unique solution if and only if* (3.7) *has a unique solution.*

Proof. First, we observe that by Theorem 4.2, if z is a solution of (3.7), then $\sigma = F'^{-1}(g_z)$ is a solution of (3.2), where $g_z = \sum_{i=1}^{n-m} z_i h_{i+m}$. Conversely, if σ is a solution of (3.2), then there is a $z \in R^{n-m}$ with $g_z = F'(\sigma)$, and, by Theorem 4.3, z satisfies (4.4). Since F' is strictly monotone while $\{h_{i+m}\}_1^{n-m}$ are linearly independent, these conditions are also sufficient for z to be a solution of (3.7). We have established a one-to-one correspondence between the solutions of (3.2) and (3.7). ▮

Now we can state a uniqueness theorem which can be applied when (4.2) and (4.3) are satisfied.

THEOREM 6.2. *Suppose $F \in \mathscr{F}$ and that* (4.2) *and* (4.3) *hold. Suppose that $F^{*''}(v) > 0$ for all $v \in F'D_F$. Then* (1.2) *has exactly one solution.*

Proof. The existence of a solution was established in Theorem 4.2. Now since ψ is concave, (4.2) implies that a sufficient condition for uniqueness in problem (3.7) is that

$$y^T H(z)\, y < 0 \qquad \text{for all } y \in \mathbb{R}^{n-m},\ y \neq 0, \tag{6.1}$$

should hold for all $z \in \mathbb{R}^{n-m}$ with $g_z \in C(I; F'D_F)$, where $H(z) = (H_{ij}(z))_{i,j=1}^{n-m}$, and

$$H_{ij}(z) = (\partial/\partial z_i)(\partial\Psi/\partial z_j)(z) = -\int_a^b F^{*''}(g_z(t))\, h_{i+m}(t)\, h_{j+m}(t)\, dt.$$

Now we can also write (6.1) as

$$\int_a^b F^{*''}(g_z(t))\, g_y^2(t)\, dt > 0 \qquad \text{for all} \quad y \in \mathbb{R}^{n-m}, \quad y \neq 0. \tag{6.2}$$

Thus the condition $F^{*''}(v) > 0$ for all $v \in F'D_F$ implies (6.1), which in turn implies uniqueness for (3.7). Lemma 6.1 then gives uniqueness for (3.2), and the totality of Λ over N_L gives uniqueness for (2.1). ▮

There are some interesting constrained minimization problems involving splines where (4.2) and (4.3) are not satisfied or where $F^{*''}$ does not satisfy the hypothesis of Theorem 6.2. The following theorem is often applicable.

THEOREM 6.3. *Let F belong to the class*

$$\mathscr{F}_0 = \{F \in \mathscr{F} : D_F = \mathbb{R}, F'D_F = \mathbb{R}, F(0) = 0, F \textit{ is symmetric, and either } F'(t)/t \textit{ or } F'^{-1}(t)/t \textit{ is monotone increasing on } (0, \infty)\}. \tag{6.3}$$

Then (3.2) *has at most one solution.*

Proof. First, we observe some properties of functions $F \in \mathscr{F}_0$. Let

$$\begin{aligned} G &= F, && \text{when } F'(t)/t \text{ is monotone increasing,} \\ &= F^*, && \text{when } F'^{-1}(t)/t \text{ is monotone increasing.} \end{aligned}$$

Then for all $s, t \in (0, \infty)$ with $t \geqslant s$,

$$G'(s + t) - G'(t) \geqslant G'(s); \tag{6.4}$$

$$G(s)/2 + G(t)/2 - G((s + t)/2) \geqslant G((t - s)/2). \tag{6.5}$$

Indeed, since $G'(s + t)/(s + t) \geqslant G'(t)/t$, we obtain $G'(s + t) \geqslant (s + t) G'(t)/t$, so that $G'(s + t) - G'(t) \geqslant sG'(t)/t \geqslant G'(s)$. This is just (6.4). Now using (6.4), we easily obtain

$$\begin{aligned} &G(s)/2 + G(t)/2 - G((s + t)/2) \\ &\quad = \int_0^{(t-s)/2} [G'(2u + s) - G'(u + s)]\, du \geqslant \int_0^{(t-s)/2} G'(u)\, du \\ &\quad = G((t - s)/2), \end{aligned}$$

which is (6.5).

Now to prove the theorem, we begin with the case where $F'(t)/t$ is monotone increasing. Suppose σ_1 and σ_2 are two solutions of (3.2). Then since V and ρ_F are both convex, $(\sigma_1 + \sigma_2)/2$ is also a solution of (3.2). But then, using the symmetry of F for the last inequality, we obtain

$$0 = \rho_F(\sigma_1)/2 + \rho_F(\sigma_2)/2 - \rho_F((\sigma_1 + \sigma_2)/2) \geqslant \rho_F(| \sigma_1 - \sigma_2 |/2).$$

This implies $\sigma_1 = \sigma_2$ a.e., since $F \in \mathscr{F}_0$ assures that F is positive on all of $(0, \infty)$.

The case where $F'^{-1}(t)/t$ is monotone increasing is similar, except we must now consider the dual problem. Suppose that both $\tilde{z}$ and $\hat{z}$ are solutions of (3.7). Then

$$\begin{aligned} 0 &= -\Psi(\tilde{z})/2 - \Psi(\hat{z})/2 + \Psi((\tilde{z} + \hat{z})/2) \\ &= \rho_{F^*}(g_{\tilde{z}})/2 + \rho_{F^*}(g_{\hat{z}})/2 - \rho_{F^*}((g_{\tilde{z}} + g_{\hat{z}})/2) \geqslant \rho_{F^*}(| g_{\tilde{z}} - g_{\hat{z}} |/2). \end{aligned}$$

As before this implies that $g_{\tilde{z}} = g_{\hat{z}}$ a.e., and thus that $z = z$. We conclude that (3.7) has a unique solution, and Lemma 6.1 implies that (3.2) must also have at most one solution. ∎

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